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Singularity Time Scale of the Kardar–Parisi–Zhang Equation in 2+1 Dimensions

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A master equation for the Kardar–Parisi–Zhang (KPZ) equation in 2+1 dimensions is developed. In the fully nonlinear regime we determine the finite time scale of the singularity formation in terms of the characteristics of forcing. The exact probability density function of the one point height field is obtained correspondingly.

KEY WORDS: Surface growth; Kardar–Parisi–Zhang equation; probability density function.

1. INTRODUCTION

Due to the technical importance and fundamental interest, a great deal of efforts have been devoted to the understanding of the mechanism of thin-film growth and the kinetic roughening of growing surfaces in various growth techniques. Analytical and numerical treatments of simple growth models suggest, quite generally, the height fluctuations have a self-similar character and their average correlations exhibit a dynamical scaling form. Numerous theoretical models have been proposed, of which the simplest nontrivial example is the Kardar–Parisi–Zhang equation[(1–7)]

$$\frac{\partial h}{\partial t} - \frac{\alpha}{2} (\nabla h)^2 = \nu \nabla^2 h + f(x, y, t). \tag{1}$$

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where $h(\mathbf{x}, \mathbf{t})$ specifies the height of the surface at point $\mathbf{x}$ and f is a zero-mean, statistically homogeneous, white in time Gaussian process with covariance $\langle f(\mathbf{x}, t)\, f(\mathbf{x}', t')\rangle = 2D_0\delta(t-t')\, D(\mathbf{x}-\mathbf{x}')$. Typically the spatial correlation of forcing is considered to be as a delta function, mimicking the short range correlation. Here we consider the spatial correlation as $D(\mathbf{x}-\mathbf{x}') = \frac{1}{\pi\sigma^2}\exp(-\frac{(\mathbf{x}-\mathbf{x}')^2}{\sigma^2})$, where σ is the spatial correlation length of random force. When σ is much less than the system size L, i.e., $\sigma \ll L$, the model represents a short range character for the forcing. The same equation is believed to describe the statistics of directed polymer in a random medium,[8] where h has the meaning of the free energy. The mean value of force on the interface is not essential and can be removed by a simple shift in h. Every term in the Eq. (1) involves a specific physical phenomenon contributing to the surface evolution. The parameters ν, α, and D_0 (and σ) are describing surface relaxation, lateral growth and the noise strength, respectively. The KPZ equation has the property of instability of local minima, which means that there is a competition between the smoothing effect of diffusion (the Laplacian operator), and the enhancement of non-zero slopes. The resulting surface $h(\mathbf{x}, t)$ could have very complicated structures. It is useful to rescale the KPZ equation using $h' = h/h_0$, $\mathbf{r}' = \mathbf{r}/\mathbf{r_0}$ and $t' = t/t_0$. If we let $h_0 = (\frac{D_0}{\nu})^{1/2}$, $t_0 = \frac{r_0^2}{\nu}$, where r_0 is a characteristic length, we can eliminate all of the parameters from the equation expect for the coupling constant $g = \frac{\alpha^2 D_0}{\nu^3}$. The limit $g \to \infty$ or $\nu \to 0$ is known as strong coupling limit.

Let us consider a substrate of size L and define the mean height of growing film $\bar{h}$ and its roughness w by: $\bar{h}(L, t) = \frac{1}{L}\int_{-L/2}^{L/2} dx\, h(x, t)$ and $w(L, t) = (\langle(h-\bar{h})^2\rangle)^{1/2}$, where $\langle\cdots\rangle$ denotes an averaging over different realizations of the noise (samples). Starting from a flat interface (one of the possible initial conditions), it was conjectured by Family and Vicsek that a scaling of space by factor b and of time by a factor b^z (z is the dynamical scaling exponent), re-scales the roughness w by factor b^χ as follows: $w(bL, b^z t) = b^\chi w(L, t)$, which implies that $w(L, t) = L^\chi f(t/L^z)$. If for large t and fixed L ($t/L^z \to \infty$), w saturates then $f(x) \to$ const., as $x \to \infty$. However, for fixed large L and $1 \ll t \ll L^z$, one expects that correlations of the height fluctuations are set up only within a distance $t^{1/z}$ and thus must be independent of L. This implies that for $x \ll 1$, $f(x) \sim x^\beta$ with $\beta = \chi/z$. Thus dynamic scaling postulates that, $w(L, t) \sim t^\beta$ for $1 \ll t \ll L^z$ and $\sim L^\chi$ for $t \gg L^z$. The roughness exponent χ and the dynamic exponent z characterize the self-affine geometry of the surface and its dynamics, respectively. Galilean invariance implies the relation $\chi + z = 2$ independent of dimension. The critical exponents of the strong-coupling regime are only known in $1+1$ dimensions and their values in higher dimensions as well as properties of the roughening transition have been known only numerically and by the various approximative schemes. In the one-dimensional substrates a fluctuation-dissipation theorem yields exactly $z = 3/2$, $\chi = 1/2$, $\beta = 1/3$ (see also ref. 15–16).

As mentioned above the limit $g \to \infty$ is known as the strong coupling limit. Despite intense effort in the recent years, the properties of the strong-coupling phase are rather poorly understood. The KPZ equation is a nonlinear partial differential equation, so it includes the possibility of forming singularities in finite time. This means that height-field (its gradients) become singular or at least non-smooth, at points, lines or even complex manifolds. Near singularities, the microscopic structure re-emerges, as the height gradient changes over arbitrarily small distances. Eventually, the singularity is cut-off by some microscopic length scale such as the distance between molecules. The most fundamental question is whether the microscopic structure becomes relevant for features of the height fluctuation much larger than microscopic ones. If this is the case, the continuum description is no longer self-consistent, but has to be supplemented by microscopic information. One of the important category of physical singularities are those which exist for a period of time, being either stationary or moving about in space like the classical example of sharp valley in the KPZ equation or shock wave in the Burgers equation. At finite viscosity, a sharp valley or shock wave is not a true singularity, but maintains a finite width δ determined by the ratio of the viscosity and the nonlinearity strength. What is important is thus the fact that the solution remains consistent as δ goes to zero. Indeed the dissipation inside the shock remains finite in this limit, so on scales much larger than δ the dynamics of height field is the same as if δ were zero.[10]

Let us summarize the properties of KPZ equation in one and two spatial dimensions in the strong coupling limit. The main properties of the KPZ equation are as follows: (i) In the limit of $\nu \to 0$ the unforced KPZ equation develops singularities for the given dimension. In one spatial dimension it develops sharp valleys in the finite time, see Fig. 1. The geometrical picture in Fig. 1 consists of a collection of sharp valleys intervening a series of hills in the stationary state. Obviously the height itself is a continuous quantity while in the sharp valleys the first derivative is discontinuous. Higher derivatives are singular *on the sharp valleys* and only can be described as distributions.[11] Indeed in one dimension we can characterize jth valley with four quantities, its location y_j, its gradients at y_{j0+} (i.e., u_+), y_{j0-} (i.e., u_-) and its height measured from the $\bar{h}$, i.e., $\tilde{h}_{vj}$. Instead of u_+ and u_- it is useful to define the quantities $\bar{u} = \frac{u_+ + u_-}{2}$ and $s = s(t) = u_+ - u_-$. The quantity s is known as the strength of sharp valley. Through elaborating the stochastic dynamical equations it comes out that the sharp valley characteristics are correlated to each other and forcing.[11]

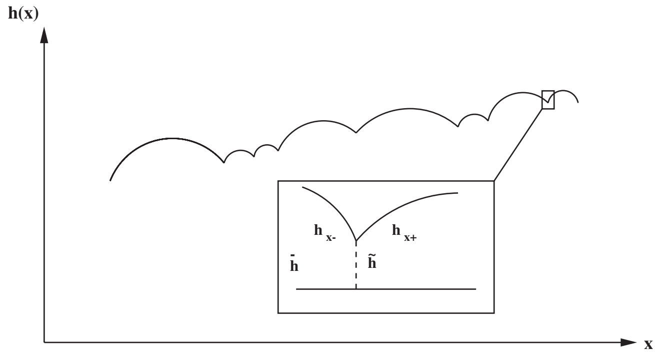

Fig. 1. The sharp valley solutions in KPZ equation are demonstrated while the variables characterising the vallies, namely h_{x-}, h_{x+} and $\bar{h}$ are shown.[11]

In two spatial dimensions KPZ equation develops three types of singularities in the finite time. The first singularities are finite sharp valley lines (shock lines) across which the height gradients is discontinuous, see Fig. 2. The second type is the end point of the sharp valley lines which separates the regular points and singular region and is called a kurtoparaboic point. As time goes these sharp valley lines hit each other and crossing point of two valley lines (shock lines) produces a valley node (shock node). Generically kurtoparabolic points disappear at large times and only a network of sharp valley lines survives.[9, 10, 13] A complete classification of the singularities of KPZ (Burgers) equation in two and three dimensions, by considering the metamorphoses of singularities as time elapses, has been done in ref. 12.

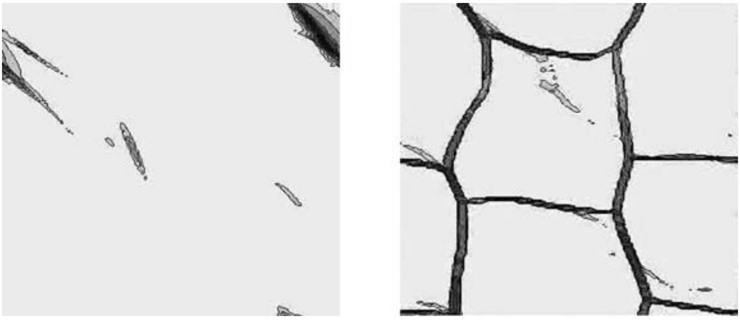

Fig. 2. Different time snapshots of two dimensional gradient configuration within system size, i.e., $-\partial_x h$ and $-\partial_y h$. In the solid lines the $h(x, y)$ is not differentiable and for the regions between the sharp-valley lines the height field is smooth function.[14]

(ii) For white in time and smooth in space forcing, also the forced KPZ equation develops singulartities in the strong coupling limit and in the finite time. In one spatial dimension the statistical properties of height fluctuations in the stationary state are determined by statistics of characteristics of sharp valleys. For instance it is shown that to leading order and for $L \gg \sigma$ the moments $S_n = \langle (h - \bar{h})^n \rangle$ scales as $S_n = A_n L^{n/2}$, where A_n depends on h_{vj}, $\bar{u}$ and s, which are defined on sharp-valleys.[11] In two spatial dimensions the sharp valley lines (cusp lines) are smooth curves and in the stationary state the sharp valley lines produces a curvilinear hexagonal lattice and therefore one finds an hexagonal tiling of singularity lines[13] (see also Fig. 2).

Here we shall adapt the master equation approach which enables us to investigate the KPZ equation in the strong-coupling (or zero tension) limit.[11] This approach enables us to determine the time scale t_c that the forced KPZ equation produces singularities. It is one of advantage of this method that all the nonlinearities due to the nonlinear term $\frac{\alpha}{2}(\nabla h)^2$ can be written in a closed form. When σ is finite, the very existence of the nonlinear term in the KPZ equation leads to the development of the singularities in a *finite time* and in the strong coupling limit ($g \to \infty$ or $\nu \to 0$). Therefore one would distinguish between different time regimes before and after the sharp valley formation, i.e., t_c. Starting from a flat initial condition, i.e., $h(x, 0) = 0$, $h_x = u(x, 0) = 0$ and $h_y = v(x, 0) = 0$, which its evolution is given by the KPZ equation with $\nu \to 0$, we know that after a finite time (t_c) the derivatives of function $h(x, t)$ becomes singular. Before creation of singularites one can disregard the diffusion term in the strong coupling limit, i.e., $\nu \to 0$. After this time scale the diffusion term is important. In this paper we present an exact solution of the model in $2+1$ dimensions in the strong coupling limit. Using the master equation method we find the time scale t_c and show that it can be expressed in terms of the characteristics of forcing and coefficient of the nonlinear term α as, $t_c = \frac{1}{4}\left(\frac{\pi}{8\alpha^2 D_0}\right)^{1/3} \sigma^2$. Also we derive the time dependence of the moments $\langle (h - \bar{h})^n \rangle$ for time scales before the creation of singularities.

Now let us consider the Kardar–Parizi–Zhang equation in $2+1$ dimension. Defining $h_x(x, y, t) = u(x, y, t)$, $h_y(x, y, t) = v(x, y, t)$, and Θ as:

$$\Theta = \exp(-i\lambda(h(x, y, t) - \bar{h}(t)) - i\mu_1 u(x, y, t) - i\mu_2 v(x, y, t)). \qquad (2)$$

enable us to express the generating function as $Z(\lambda, \mu_1, \mu_2, x, y, t) = \langle \Theta \rangle$. Using the KPZ equation and neglecting the viscosity term, in the limit of $\nu \to 0$ before the creation of singularities, we get:

$$Z_t = i\gamma(t)\,\lambda Z + i\lambda\frac{\alpha}{2}Z_{\mu_1\mu_1} + i\lambda\frac{\alpha}{2}Z_{\mu_2\mu_2} - i\alpha\lambda Z_{\mu_1\mu_1}$$
$$- i\alpha\lambda Z_{\mu_2\mu_2} - \alpha\langle u_x\Theta\rangle - \alpha\langle v_y\Theta\rangle$$
$$- \lambda^2 k(0,0)\,Z + \mu_1^2 k''(0,0)\,Z + \mu_2 k''(0,0)\,Z \tag{3}$$

where $\gamma(t) = \bar{h}(t) = \frac{\alpha}{2}\langle u^2 + v^2\rangle$, $k(\mathbf{x}-\mathbf{x}') = 2D_0 D(\mathbf{x}-\mathbf{x}')$, $k''(0,0) = k_{xx}(0,0) = k_{yy}(0,0)$ and $\frac{\alpha}{2}\langle u^2\rangle = \frac{\alpha}{2}\langle v^2\rangle = -\alpha k''(0,0)\,t$ and we have used the homogeneity condition, i.e., $Z_x = Z_y = 0$. The term $\langle u_x\Theta\rangle$ and $\langle v_y\Theta\rangle$ and can be evaluated as follows $\langle u_x\Theta\rangle = -i\frac{\lambda}{\mu_1}Z_{\mu_1} - \frac{\mu_2}{\mu_1}\langle v_x\Theta\rangle$, and $\langle v_y\Theta\rangle = -i\frac{\lambda}{\mu_2}Z_{\mu_2} - \frac{\mu_1}{\mu_2}\langle u_y\Theta\rangle$. In appendix we have proved that $\langle u_y\Theta\rangle = \langle v_x\Theta\rangle = 0$, so the equation governing the time evolution of Z becomes

$$Z_t = i\gamma(t)\,\lambda Z - i\lambda\frac{\alpha}{2}Z_{\mu_1\mu_1} - i\lambda\frac{\alpha}{2}Z_{\mu_2\mu_2}$$
$$+ i\alpha\frac{\lambda}{\mu_1}Z_{\mu_1} - i\alpha\frac{\lambda}{\mu_2}Z_{\mu_2}$$
$$- \lambda^2 k(0,0)\,Z + \mu_1^2 k''(0,0)\,Z + \mu_2 k''(0,0)\,Z. \tag{4}$$

To derive the Eq. (4) we have used the Gaussianity of the forcing statistics which help us to write its contribution in terms of generation function according to a typical trick in Gaussian random forcing. The above equation can be solved exactly with initial condition $Z(\lambda, \mu_1, \mu_2, 0) = 1$. The solution has the following form,[7, 11]

$$Z(\lambda,\mu_1,\mu_2,t) = \left(1 - \tanh^2\left(\sqrt{2ik''(0,0)\,\alpha\lambda}\;t\right)\right)^2$$
$$\times \exp\Bigg[-\frac{5}{4}\ln\left(1 - \tanh^4\left(\sqrt{2ik''(0,0)\,\alpha\lambda}\;t\right)\right)$$
$$+ \frac{5}{2}\tanh^{-1}\left(\tanh^2\left(\sqrt{2ik''(0,0)\,\alpha\lambda}\;t\right)\right) - \lambda^2 k(0,0)\,t$$
$$- \frac{1}{8}\ln^2\left(\frac{1 - \tanh\left(\sqrt{2ik''(0,0)\,\alpha\lambda}\;t\right)}{1 + \tanh\left(\sqrt{2ik''(0,0)\,\alpha\lambda}\;t\right)}\right)$$
$$- \frac{1}{2}i(\mu_1^2 + \mu_2^2)\sqrt{\frac{2ik''(0,0)}{\alpha\lambda}}\tanh\left(\sqrt{2ik''(0,0)\,\alpha\lambda}\;t\right)\Bigg]. \tag{5}$$

Using the generation function we can derive the probability distribution function (PDF) of height fluctuation by inverse Fourier transformation. Expanding the solution of the generating function in powers of λ all the moments $\langle (h-\bar{h})^n \rangle$ can be derived. For instance the first fifth moments before sharp valley formation are:

$$\langle \tilde{h}^2 \rangle = \left(\frac{k^2(0,0)}{\alpha k''(0,0)} \right)^{2/3} \left[-\frac{2}{3}\left(\frac{t}{t_*}\right)^4 + 2\frac{t}{t_*} \right]$$

$$\langle \tilde{h}^3 \rangle = -\frac{48}{45}\left(\frac{k^2(0,0)}{\alpha k''(0,0)} \right)\left(\frac{t}{t_*}\right)^6$$

$$\langle \tilde{h}^4 \rangle = \left(\frac{k^2(0,0)}{\alpha k''(0,0)} \right)^{4/3} \left[-\frac{44}{35}\left(\frac{t}{t_*}\right)^8 - 8\left(\frac{t}{t_*}\right)^5 + 12\left(\frac{t}{t_*}\right)^2 \right] \tag{6}$$

$$\langle \tilde{h}^5 \rangle = -\left(\frac{k^2(0,0)}{\alpha k''(0,0)} \right)^{5/3} \left[\frac{1216}{945}\left(\frac{t}{t_*}\right)^{10} + \frac{64}{3}\left(\frac{t}{t_*}\right)^7 \right]$$

where $t_* = (\frac{k(0,0)}{\alpha^2 k''^2(0,0)})^{1/3} = (\frac{\pi}{8\alpha^2 D_0})^{1/3}\,\sigma^2$. The important content of the exact expressions derived above is that through them the time scale of singularities formation can be found. We have found the time scale t_c by checking the positivity condition of PDF, i.e., $P(h-\bar{h}, t) \geqslant 0$. The PDF of height fluctuation $p(h-\bar{h}, t)$ can be derived by inverse Fourier transformation. The positivity of PDF means that all the even moments of $\langle (h-\bar{h})^n \rangle$ must be positive. In fact the above moment relations indicate that different even order moments become *negative* in some distinct characteristic time scales. Closer looking in the even moment relations reveals that the higher the moments are, the smaller their characteristic time scales become such that asymptotically tends to $t_c = \frac{1}{4} t_*$ for very large even moments. It can be shown that after time scale t_c the *right tail* of the probability distribution function (PDF) of height fluctuations (i.e., $P(h-\bar{h}, t)$) is going to become negative, which is reminiscent of the singularity creation.

Let us discuss that *why* and *how* the time scale of negativity of right tail of PDF related to the time scale of singularity formation t_c. We remind that to derive the result presented in Eq. (5), for the generating function Z, we neglected the diffusion term in the limit $\nu \to 0$. Therefore, effectively we have the inviscid KPZ equation. In this limit after t_c the KPZ equation produces multi-valued solution.[10] Indeed the connection between the negativity of right tail and time scale of singularity formation is related to multi-valued solution of KPZ equation after time scale t_c. We note that the Eq. (5) has the property that $Z(0, 0, 0, t) = 1$ which means that $\int_{-\infty}^{+\infty} P(h-\bar{h}, u, v; t)\, d(h-\bar{h})\, du\, dv = 1$ for every time t. So the PDF of $h-\bar{h}$ and its derivatives is always normalizable to unity. In the limit of $\nu \to 0$

after t_c height field becomes multi-value on the valleys, which is related to the left tail of the $P(h-\bar{h})$. The multiplicity of height field on valleys, on which the height difference $h-\bar{h}$ is mostly negative, increases the probability measure in left tail of the PDF. Therefore to compensate the exceeded measure related to the multi-valued solutions the right tail of the PDF tails should become negative. The singularities in the limit $\nu \to 0$ can be constructed from multi-valued solutions of the KPZ equation with $\nu = 0$ by Maxwell cutting rule,[10] which makes the discontinuity in the derivative of height field. For time scales after the characteristic time scale t_c one should also consider the contribution of the relaxation term in the limit of vanishing diffusion coefficient in order to find a positive probability density function of height field. In other words disregarding the diffusion term in the PDF equation is valid only up to the time scales in which the singularities are developed.

Taking into account that $\alpha > 0$ and $k''(0, 0) < 0$, the odd order moments are positive in time scales before formation of singularities. It means that the probability density function $P(h-\bar{h}, t)$ in this time regime is positively skewed. Therefore the probability distribution functions of height difference has a non zero skewness as it evolves in time, at least up to the time scale where the singularities are formed. We have performed the inverse Fourier transform of the generating function numerically to get the form of the $P(h-\bar{h})$. To demonstrate the time scale of the singularity formation we have numerically sketched the PDF evolution in time, Fig. 3. As the system evolves in time, the formation of the first singularities leads to the negativity of the right tail in the PDF.

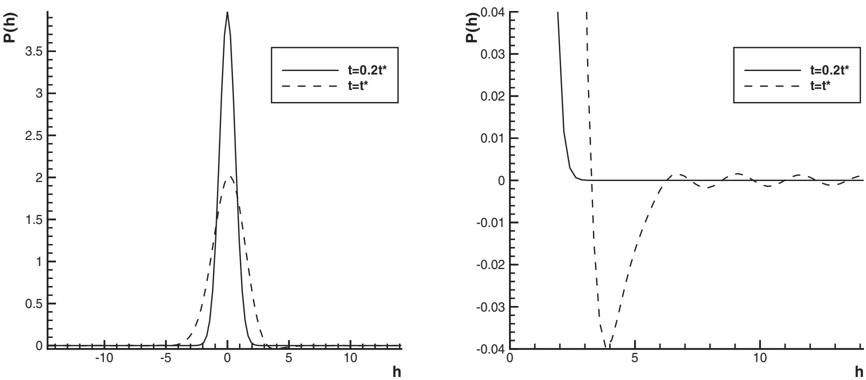

Fig. 3. In the left graph the time evolution of PDF of $h-\bar{h}$ before singularity formation at $\frac{2}{10}t_*$ and t_* is numerically obtained. Right graph shows the right tails of the PDF of $h-\bar{h}$ for $\frac{2}{10}t_*$ and t_* corresponding to before and after singularities formation which are numerically calculated.

Let us discuss about the stationary state of the 2-dimensional KPZ equation in the strong coupling limit. Using the numerical results it is shown recently that convergence to the statistical steady state is reached after a few turnover times.[13] Therefore in the stationary state the singularities will fully developed and at $t \to \infty$ we should take into account the relaxation contribution in the limit $\nu \to 0$. Contribution of the relaxation term in the rhs of the Eq. (4) is,

$$\lim_{\nu \to 0} \{-i\nu\mu_1\langle \nabla^2 u\Theta\rangle - i\nu\mu_2\langle \nabla^2 v\Theta\rangle\}. \tag{7}$$

In ref. 7 we have calculated the finite contribution of this term in $1+1$ dimensions. In $2+1$ dimension calculation of finite contribution of the relaxation term is more complex (because of complex structure of the singularities) and has been left for future works.

To summarize we obtain some results in the problem of KPZ equation in $2+1$ dimensions with a Gaussian forcing which is white in time and short range correlated in space. In the non-stationary regime when the sharp valley structures are not yet developed we find an exact form for the generating function of the joint fluctuations of height and height gradients. We determine the time scale of the sharp valley formation and the exact functional form of the time dependence in the height difference moments at any given order. We believe that the analysis followed in this paper is also suitable for the zero temperature limit in the problem of directed polymer in the random potential with short range correlations.[8] In the same direction the present method is applicable to the *strong coupling* regime of KPZ equation in higher dimensions ($d > 2$) which is definitely an important step. This methodology is one of the rare ways to tackle the strong coupling regime of KPZ.

APPENDIX

In this appendix we present another way of deriving the height moments by introducing the second spatial derivatives of height field and also prove that $\langle u_y\Theta\rangle = \langle v_x\Theta\rangle = 0$. Defining $\Theta(\lambda, \mu_1, \mu_2, \eta_1, \eta_2, \eta_3, x, y, t)$ as

$$\begin{aligned}\Theta(\lambda, \mu_1, \mu_2, \eta_1, \eta_2, \eta_3, x, y, t) &= \exp\{-i\lambda(h(x,y,t)-\bar{h}(t)) - i\mu_1 u(x,y,t) - i\mu_2 v(x,y,t) \\ &\quad - i\eta_1 w(x,y,t) - i\eta_2 s(x,y,t) - i\eta_3 q(x,y,t)\},\end{aligned} \tag{8}$$

the generating function is defined as

$$Z(\lambda, \mu_1, \mu_2, \eta_1, \eta_2, \eta_3, x, y, t) = \langle \Theta \rangle, \tag{9}$$

where $u(x, y, t) = h_x(x, y, t)$, $v(x, y, t) = h_y(x, y, t)$, $w(x, y, t) = h_{xx}(x, y, t)$, $s(x, y, t) = h_{xy}(x, y, t)$ and $q(x, y, t) = h_{yy}(x, y, t)$.

The evolution of $h(x, y, t)$, $u(x, y, t)$, $v(x, y, t)$, $w(x, y, t)$, $s(x, y, t)$, $q(x, y, t)$ is given by the following equations

$$h_t = \frac{\alpha}{2}(u^2 + v^2) + f(x, y, t) \tag{10}$$

$$u_t = \alpha(uw + vs) + f_x(x, y, t) \tag{11}$$

$$v_t = \alpha(us + vq) + f_y(x, y, t) \tag{12}$$

$$w_t = \alpha(w^2 + s^2 + uw_x + vs_x) + f_{xx}(x, y, t) \tag{13}$$

$$s_t = \alpha(ws + qs + vs_y uw_y) + f_{xy}(x, y, t) \tag{14}$$

$$q_t = \alpha(s^2 + q^2 + us_y + vq_y) + f_{yy}. \tag{15}$$

It follows from the above equations that the generating function Z is the solution of the following equation

$$\begin{aligned} Z_t = {} & i\gamma(t)\,\lambda Z - i\lambda\frac{\alpha}{2} Z_{\mu_1\mu_1} - i\lambda\frac{\alpha}{2} Z_{\mu_2\mu_2} + i\alpha Z_{\mu_1 x} + i\alpha Z_{\mu_2 y} \\ & - i\alpha Z_{\eta_1} - i\alpha Z_{\eta_3} + i\alpha\eta_1 Z_{\eta_1\eta_1} + i\alpha\eta_1 Z_{\eta_2\eta_2} \\ & + i\alpha\eta_2 Z_{\eta_1\eta_2} + i\alpha\eta_2 Z_{\eta_2\eta_3} + i\alpha\eta_3 Z_{\eta_2\eta_2} + i\alpha\eta_3 Z_{\eta_3\eta_3} \\ & - i\lambda\langle f(x, y, t)\,\Theta\rangle - i\mu_1\langle f_x(x, y, t)\,\Theta\rangle - i\mu_2\langle f_y(x, y, t)\,\Theta\rangle \\ & - i\eta_1\langle f_{xx}(x, y, t)\,\Theta\rangle - i\eta_2\langle f_{xy}(x, y, t)\,\Theta\rangle - i\eta_3\langle f_{yy}(x, y, t)\,\Theta\rangle, \end{aligned} \tag{16}$$

in which $\gamma(t)$ is defined as $\gamma(t) = \bar{h}_t$ and the following identities have been used

$$\eta_1\langle w_x\Theta\rangle + \eta_2\langle s_x\Theta\rangle + \eta_3\langle q_x\Theta\rangle = iZ_x - i\lambda Z_{\mu_1} - i\mu_1 Z_{\eta_1} - i\mu_2 Z_{\eta_2} \tag{17}$$

$$\eta_1\langle w_y\Theta\rangle + \eta_2\langle s_y\Theta\rangle + \eta_3\langle q_y\Theta\rangle = iZ_y - i\lambda Z_{\mu_2} - i\mu_1 Z_{\eta_2} - i\mu_2 Z_{\eta_3}. \tag{18}$$

Now, using Novikov's theorem we find

$$\langle f(x, y, t)\,\Theta\rangle = -i\lambda k(0, 0)\,Z - i\eta_1 k_{xx}(0, 0)\,Z - i\eta_3 k_{xx}(0, 0)\,Z \tag{19}$$

$$\langle f_x(x, y, t)\,\Theta\rangle = i\mu_1 k_{xx}(0, 0)\,Z \tag{20}$$

$$\langle f_y(x, y, t)\,\Theta\rangle = i\mu_2 k_{xx}(0, 0)\,Z \tag{21}$$

$$\langle f_{xx}(x, y, t)\, \Theta \rangle = -i\lambda k_{xx}(0,0)\, Z - i\eta_1 k_{xxxx}(0,0)\, Z - i\eta_3 k_{xxxx}(0,0)\, Z \tag{22}$$

$$\langle f_{xy}(x, y, t)\, \Theta \rangle = -i\eta_2 k_{xxxx}(0,0)\, Z \tag{23}$$

$$\langle f_{yy}(x, y, t)\, \Theta \rangle = -i\lambda k_{xx}(0,0)\, Z - i\eta_1 k_{xxxx}(0,0)\, Z - i\eta_3 k_{xxxx}(0,0)\, Z, \tag{24}$$

where $k(x-x', y-y') = 2D_0 D(x-x', y-y')$, $k(0,0) = \frac{2D_0}{\pi\sigma^2}$, $k_{xx}(0,0) = k_{yy}(0,0) = -\frac{4D_0}{\pi\sigma^4}$ and $k_x(0,0) = k_y(0,0) = 0$. So we have

$$\begin{aligned} Z_t = {} & i\gamma(t)\, \lambda Z - i\lambda \frac{\alpha}{2} Z_{\mu_1\mu_1} - i\lambda \frac{\alpha}{2} Z_{\mu_2\mu_2} + i\alpha Z_{\mu_1 x} + i\alpha Z_{\mu_2 y} \\ & - i\alpha Z_{\eta_1} - i\alpha Z_{\eta_3} + i\alpha\eta_1 Z_{\eta_1\eta_1} + i\alpha\eta_1 Z_{\eta_2\eta_2} \\ & + i\alpha\eta_2 Z_{\eta_1\eta_2} + i\alpha\eta_2 Z_{\eta_2\eta_3} + i\alpha\eta_3 Z_{\eta_2\eta_2} + i\alpha\eta_3 Z_{\eta_3\eta_3} \\ & - \lambda^2 k(0,0)\, Z + (\mu_1^2 + \mu_2^2 - 2\lambda\eta_1 - 2\lambda\eta_3)\, k_{xx}(0,0)\, Z \\ & - (\eta_1^2 + \eta_2^2 + \eta_3^2 + 2\eta_1\eta_3)\, k_{xxxx}(0,0)\, Z. \end{aligned} \tag{25}$$

Assuming statistical homogeneity ($Z_x = 0$, $Z_y = 0$) and defining $P(\tilde{h}, u, v, w, s, q, t)$ as the joint probability density function of $\tilde{h}$, u, v, w, s, and q, one can construct the PDF as the Fourier transform of the generating function Z with respect to $\lambda, \mu_1, \mu_2, \eta_1, \eta_2, \eta_3$

$$\begin{aligned} & P(\tilde{h}, u, v, w, s, q, t) \\ & \quad = \int \frac{d\lambda}{2\pi} \frac{d\mu_1}{2\pi} \frac{d\mu_2}{2\pi} \frac{d\eta_1}{2\pi} \frac{d\eta_2}{2\pi} \frac{d\eta_3}{2\pi} \\ & \qquad \times \{\exp(i\lambda\tilde{h} + i\mu_1 u + i\mu_2 v + i\eta_1 w + i\eta_2 s + i\eta_3 q)\, Z(\lambda, \mu_1, \mu_2, \eta_1, \eta_2, \eta_3, t)\}. \end{aligned} \tag{26}$$

From the Eqs. (25) and (26) the equation governing the evolution of $P(\tilde{h}, u, v, w, s, q, t)$ can be derived, which is

$$\begin{aligned} P_t = {} & \gamma(t)\, P_{\tilde{h}} + \frac{\alpha}{2}(u^2 + v^2)\, P_{\tilde{h}} - 4\alpha w P - 4\alpha q P \\ & - \alpha w^2 P_w - \alpha q^2 P_q - \alpha s^2 P_w - \alpha s^2 P_q - \alpha w s P_s - \alpha q s P_s \\ & + k(0,0)\, P_{\tilde{h}\tilde{h}} + 2k_{xx}(0,0)\, P_{\tilde{h}w} + 2k_{xx}(0,0)\, P_{\tilde{h}q} - k_{xx}(0,0)\, P_{uu} - k_{xx}(0,0)\, P_{vv} \\ & + k_{xxxx}(0,0)\, P_{ww} + k_{xxxx}(0,0)\, P_{ss} + k_{xxxx}(0,0)\, P_{qq} + 2k_{xxxx}(0,0)\, P_{qw}. \end{aligned} \tag{27}$$

From the Eq. (27), it is easy to see that the moments $\langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3} s^{n_4} q^{n_5} \rangle$ satisfy the following equation

$$
\begin{aligned}
&\frac{d}{dt}\langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3} s^{n_4} q^{n_5} \rangle \\
&\quad = -n_0 \gamma(t) \langle \tilde{h}^{n_0-1} u^{n_1} v^{n_2} w^{n_3} s^{n_4} q^{n_5} \rangle \\
&\qquad -\frac{\alpha n_0}{2} \langle \tilde{h}^{n_0-1} u^{n_1+2} v^{n_2} w^{n_3} s^{n_4} q^{n_5} \rangle \\
&\qquad -\frac{\alpha n_0}{2} \langle \tilde{h}^{n_0-1} u^{n_1} v^{n_2+2} w^{n_3} s^{n_4} q^{n_5} \rangle \\
&\qquad -4\alpha \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3+1} s^{n_4} q^{n_5} \rangle \\
&\qquad -4\alpha \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3} s^{n_4} q^{n_5+1} \rangle \\
&\qquad +\alpha(n_3+2) \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3+1} s^{n_4} q^{n_5} \rangle \\
&\qquad +\alpha n_3 \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3-1} s^{n_4+2} q^{n_5} \rangle + \alpha(n_4+1) \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3+1} s^{n_4} q^{n_5} \rangle \\
&\qquad +\alpha(n_4+1) \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3} s^{n_4} q^{n_5+1} \rangle + \alpha(n_5+2) \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3} s^{n_4} q^{n_5+1} \rangle \\
&\qquad +n_5(n_5-1)\, k_{xxxx}(0,0) \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3} s^{n_4} q^{n_5-2} \rangle \\
&\qquad +2 n_3 n_5 k_{xxxx}(0,0) \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3-1} s^{n_4} q^{n_5-1} \rangle \\
&\qquad +2 n_0 n_3 k_{xx}(0,0) \langle \tilde{h}^{n_0-1} u^{n_1} v^{n_2} w^{n_3-1} s^{n_4} q^{n_5} \rangle \\
&\qquad +2 n_0 n_5 k_{xx}(0,0) \langle \tilde{h}^{n_0-1} u^{n_1} v^{n_2} w^{n_3} s^{n_4} q^{n_5-1} \rangle \\
&\qquad -n_1(n_1-1)\, k_{xx}(0,0) \langle \tilde{h}^{n_0} u^{n_1-2} v^{n_2} w^{n_3} s^{n_4} q^{n_5} \rangle \\
&\qquad -n_2(n_2-1)\, k_{xx}(0,0) \langle \tilde{h}^{n_0} u^{n_1} v^{n_2-2} w^{n_3} s^{n_4} q^{n_5} \rangle \\
&\qquad +n_3(n_3-1)\, k_{xxxx}(0,0) \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3-2} s^{n_4} q^{n_5} \rangle \\
&\qquad +n_4(n_4-1)\, k_{xxxx}(0,0) \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3} s^{n_4-2} q^{n_5} \rangle \\
&\qquad +\alpha n_5 \langle \tilde{h}^{n_0} u^{n_1} v^{n_2} w^{n_3} s^{n_4+2} q^{n_5-1} \rangle + n_0(n_0-1)\, k(0,0) \langle \tilde{h}^{n_0-2} u^{n_1} v^{n_2} w^{n_3} s^{n_4} q^{n_5} \rangle.
\end{aligned}
\tag{28}
$$

By substituting different sort of values for n_0, n_1, n_2, n_3, n_4, and n_5 we can find some coupled differential equations for different moments. For example, if we take $n_0 = n_1 = \cdots = n_5 = 0$ we have

$$\langle w \rangle + \langle q \rangle = 0 \tag{29}$$

or

$$\langle \nabla \cdot \mathbf{u} \rangle = 0 \tag{30}$$

where this is the same as the statistical hemogeneity condition. For $n_0 = 1, n_1 = n_2 = n_3 = n_4 = n_5 = 0$ we find

$$\langle \tilde{h}w \rangle + \langle \tilde{h}q \rangle = -\langle u^2 \rangle - \langle v^2 \rangle \tag{31}$$

and for $n_0 = 0, n_1 = 1, n_2 = n_3 = n_4 = n_5 = 0$ we find

$$\langle uw \rangle + \langle uq \rangle = 0, \tag{32}$$

and if we assume the statistical hemogeneity we have

$$\langle uw \rangle = \langle uu_x \rangle = \tfrac{1}{2} \langle u^2 \rangle_x = 0. \tag{33}$$

So $\langle uw \rangle = 0$ and then $\langle uq \rangle = 0$, also for $n_0 = n_1 = 0,\ n_2 = 1,\ n_3 = n_4 = n_5 = 0$ we find $\langle vq \rangle = 0$, $\langle vw \rangle = 0$.

For $n_0 = 0, n_1 = 2, n_2 = n_3 = n_4 = n_5 = 0$ we find

$$\frac{d}{dt} \langle u^2 \rangle = -\alpha \langle u^2 w \rangle - \alpha \langle u^2 q \rangle - 2k_{xx}(0, 0) \tag{34}$$

and also we have $\langle u^2 w \rangle = \tfrac{1}{3} \langle u^3 \rangle_x = 0$ by statistical hemogeneity and also $\langle u^2 q \rangle = \langle u^2 v \rangle_y - 2\langle uvs \rangle$, $\langle u^2 v \rangle_y = 0$, so $\langle u^2 q \rangle = -2\langle uvs \rangle$ so that we have

$$\frac{d}{dt}\langle u^2 \rangle = 2\alpha \langle uvs \rangle - 2k_{xx}(0, 0). \tag{35}$$

The corresponding differential equation for $\langle uvs \rangle$ is

$$\frac{d}{dt}\langle uvs \rangle = 0. \tag{36}$$

If we assume that at $t = 0$ the surface is flat so all the moments at $t = 0$ are zero, so that $\langle uvs \rangle = 0$ and then we find

$$\langle u^2 \rangle = -2k_{xx}(0, 0)\, t. \tag{37}$$

Similar calculations give

$$\langle v^2 \rangle = -2k_{xx}(0, 0)\, t. \tag{38}$$

By focusing on the differential equation of the PDF it is deduced that this equation is invariant under $u \to -u$ and $v \to -v$ which is a consequence of the inversion and reflection symmetry of the KPZ equation. So it will result in the vanishing of the u and v odd moments

$$\langle u^{2k+1} \rangle = \langle v^{2k+1} \rangle = 0. \tag{39}$$

Using Eq. (28) the even moments of u and v can be calculated. For example, for $\langle u^4 \rangle$ we have

$$\frac{d}{dt}\langle u^4 \rangle = -\alpha\langle u^4 w\rangle - \alpha\langle u^4 q\rangle - 12k_{xx}(0)\langle u^2\rangle. \tag{40}$$

On the other hand $\langle u^4 w\rangle$ can be written as

$$\langle u^4 u_x\rangle = \tfrac{1}{5}\langle u^5\rangle_x, \tag{41}$$

which is zero by homogeneity, also for $\langle u^4 q\rangle$ we have

$$\langle u^4 v_y\rangle = \langle u^4 v\rangle_y - 4\langle u^3 vs\rangle = -4\langle u^3 vs\rangle \tag{42}$$

in which $\langle u^4 v\rangle_y$ is zero by homogeneity. The differential equation for $\langle u^3 vs\rangle$ is

$$\frac{d}{dt}\langle u^3 vs\rangle = 0, \tag{43}$$

so $\langle u^3 vs\rangle = 0$. Therefore, it is obtained

$$\langle u^4\rangle = 12k_{xx}^2(0, 0)\, t^2, \tag{44}$$

and we have the same for $\langle v^4\rangle$. Also it can be found that $\langle u^2 v^2\rangle = 4k_{xx}^2(0, 0)\, t^2$. By continuing the above method all the moments $\langle u^n v^m\rangle$ can be deduced.

On the other hand, the above calculations show that all the mixed moments $\langle u^n v^m s\rangle$ are zero. In the following some of the moments of u and v and their combinations have been given

$$\langle u^6\rangle = \langle v^6\rangle = -120k_{xx}^3(0, 0)\, t^3 \tag{45}$$

$$\langle u^2 v^4\rangle = \langle u^4 v^2\rangle = -24k_{xx}^3(0, 0)\, t^3 \tag{46}$$

$$\langle u^8\rangle = \langle v^8\rangle = 1680k_{xx}^4(0, 0)\, t^4 \tag{47}$$

$$\langle u^6 v^2\rangle = \langle u^2 v^6\rangle = 240k_{xx}^4(0, 0)\, t^4 \tag{48}$$

$$\langle u^4 v^4\rangle = 144k_{xx}^4(0, 0)\, t^4, \text{ etc.} \tag{49}$$

We aim to calculate all the momentes of $\tilde{h} = h - \bar{h}$. Putting $n_0 = 2$, $n_1 = n_2 = \cdots = 0$ in the Eq. (28), we find

$$\frac{d}{dt}\langle \tilde{h}^2 \rangle = -\alpha\langle \tilde{h}u^2 \rangle - \alpha\langle \tilde{h}v^2 \rangle - \alpha\langle \tilde{h}^2 w \rangle - \alpha\langle \tilde{h}^2 q \rangle + 2k(0,0). \tag{50}$$

We know that $\langle \tilde{h}^2 w \rangle = \langle \tilde{h}^2 u \rangle_x - 2\langle \tilde{h}u^2 \rangle$ and because of the homogeneity $\langle \tilde{h}^2 u \rangle_x = 0$, so we have

$$\langle \tilde{h}^2 w \rangle = -2\langle \tilde{h}u^2 \rangle, \tag{51}$$

and also

$$\langle \tilde{h}^2 q \rangle = -2\langle \tilde{h}v^2 \rangle, \tag{52}$$

then

$$\frac{d}{dt}\langle \tilde{h}^2 \rangle = \alpha(\langle \tilde{h}u^2 \rangle + \langle \tilde{h}v^2 \rangle) + 2k(0,0). \tag{53}$$

As it can be seen from Eq. (53) we need the moments $\langle \tilde{h}u^2 \rangle$ and $\langle \tilde{h}v^2 \rangle$ to find $\langle \tilde{h}^2 \rangle$, which the related differential equations are

$$\frac{d}{dt}\langle \tilde{h}u^2 \rangle = -\gamma(t)\langle u^2 \rangle - \frac{\alpha}{2}\langle u^4 \rangle - \frac{\alpha}{2}\langle u^2 v^2 \rangle - \alpha\langle \tilde{h}u^2 w \rangle - \alpha\langle \tilde{h}u^2 q \rangle \tag{54}$$

$$\frac{d}{dt}\langle \tilde{h}v^2 \rangle = -\gamma(t)\langle v^2 \rangle - \frac{\alpha}{2}\langle v^4 \rangle - \frac{\alpha}{2}\langle u^2 v^2 \rangle - \alpha\langle \tilde{h}v^2 w \rangle - \alpha\langle \tilde{h}v^2 q \rangle. \tag{55}$$

By using of the statistical homogeneity the last two terms of the above equations can be converted as

$$\begin{aligned}
\langle \tilde{h}u^2 w \rangle &= -\tfrac{1}{3}\langle u^4 \rangle \\
\langle \tilde{h}v^2 q \rangle &= -\tfrac{1}{3}\langle v^4 \rangle \\
\langle \tilde{h}u^2 q \rangle &= -\langle u^2 v^2 \rangle - 2\langle \tilde{h}uvs \rangle \\
\langle \tilde{h}v^2 w \rangle &= -\langle u^2 v^2 \rangle - 2\langle \tilde{h}uvs \rangle.
\end{aligned} \tag{56}$$

The above relations result in

$$\begin{aligned}
\frac{d}{dt}(\langle \tilde{h}u^2 \rangle + \langle \tilde{h}v^2 \rangle) = -\gamma(t)(\langle u^2 + v^2 \rangle) - \frac{\alpha}{6}(\langle u^4 + v^4 \rangle) \\
+ \alpha\langle u^2 v^2 \rangle + 4\alpha\langle \tilde{h}uvs \rangle.
\end{aligned} \tag{57}$$

Obtaining the differential equation for $\langle \tilde{h}uvs \rangle$ we get

$$\frac{d}{dt}\langle \tilde{h}uvs \rangle = 0, \tag{58}$$

which results in $\langle \tilde{h}uvs \rangle = 0$. Also it is easy to see that all the moments $\langle \tilde{h}^n u^m v^p s \rangle$ are zero. This fantastic result helps us to find all the $\langle \tilde{h}^n \rangle$ moments. Now by substituting $\gamma(t) = \bar{h}_t = \frac{\alpha}{2}(\langle u^2 + v^2 \rangle) = -2\alpha k_{xx}(0,0)\, t$, $\langle u^4 \rangle$, $\langle v^4 \rangle$, and $\langle u^2 v^2 \rangle$ in Eq. (57) we obtain

$$\langle \tilde{h}u^2 \rangle + \langle \tilde{h}v^2 \rangle = -\tfrac{8}{3}\,\alpha k_{xx}^2(0,0)\, t^3, \tag{59}$$

which finally gives

$$\langle \tilde{h}^2 \rangle = -\tfrac{2}{3}\,\alpha^2 k_{xx}^2(0,0)\, t^4 + 2k(0,0)\, t. \tag{60}$$

Now we begin to calculate the moment $\langle \tilde{h}^3 \rangle$. Inserting $n_0 = 3$, $n_1 = n_2 = \cdots = n_5 = 0$ in Eq. (28) we get

$$\frac{d}{dt}\langle \tilde{h}^3 \rangle = -3\gamma(t)\langle \tilde{h}^2 \rangle - \frac{3}{2}\alpha(\langle \tilde{h}^2 u^2 \rangle + \langle \tilde{h}^2 v^2 \rangle) - \alpha\langle \tilde{h}^3 w \rangle - \alpha\langle \tilde{h}^3 q \rangle, \tag{61}$$

and again by using statisiacal homogeneity it can be shown that

$$\langle \tilde{h}^3 w \rangle = -3\langle \tilde{h}^2 u^2 \rangle \tag{62}$$

$$\langle \tilde{h}^3 q \rangle = -3\langle \tilde{h}^2 v^2 \rangle. \tag{63}$$

So we have

$$\frac{d}{dt}\langle \tilde{h}^3 \rangle = -3\gamma(t)\langle \tilde{h}^2 \rangle + \frac{3}{2}\alpha(\langle \tilde{h}^2 u^2 \rangle + \langle \tilde{h}^2 v^2 \rangle). \tag{64}$$

To calculate $\langle \tilde{h}^3 \rangle$ we need $\langle \tilde{h}^2 \rangle$, $(\langle \tilde{h}^2 u^2 + \tilde{h}^2 v^2 \rangle)$. $\langle \tilde{h}^2 \rangle$ has been calculated above, so we will obtain $\langle \tilde{h}^2 u^2 + \tilde{h}^2 v^2 \rangle$ using the corresponding differential equation as follows,

$$\begin{aligned} &\frac{d}{dt}(\langle \tilde{h}^2 u^2 \rangle + \langle \tilde{h}^2 v^2 \rangle) \\ &\quad = -2\gamma(t)(\langle \tilde{h}u^2 + \tilde{h}v^2 \rangle) - \alpha(\langle \tilde{h}u^4 + \tilde{h}v^4 \rangle) - 2\alpha\langle \tilde{h}u^2 v^2 \rangle - \alpha(\langle \tilde{h}^2 u^2 w + \tilde{h}^2 v^2 w \rangle), \\ &\qquad - \alpha(\langle \tilde{h}^2 u^2 q + \tilde{h}^2 v^2 q \rangle) + 2k(0,0)(\langle u^2 + v^2 \rangle) - 4k_{xx}(0,0)\langle \tilde{h}^2 \rangle. \end{aligned} \tag{65}$$

As before, we easily get

$$\begin{aligned}
\langle \tilde{h}^2 u^2 w \rangle &= -\tfrac{2}{3}\langle \tilde{h} u^4 \rangle \\
\langle \tilde{h}^2 v^2 q \rangle &= -\tfrac{2}{3}\langle \tilde{h} v^4 \rangle \\
\langle \tilde{h}^2 u^2 q \rangle &= -2\langle \tilde{h} u^2 v^2 \rangle - 2\langle \tilde{h}^2 uvs \rangle \\
\langle \tilde{h}^2 v^2 w \rangle &= -2\langle \tilde{h} u^2 v^2 \rangle - 2\langle \tilde{h}^2 uvs \rangle .
\end{aligned} \tag{66}$$

As discussed before, it is easy to show that the moment $\langle \tilde{h}^2 uvs \rangle$ is zero. To prove this we write the corresponding differential equation

$$\frac{d}{dt}\langle \tilde{h}^2 uvs \rangle = -\alpha(\langle \tilde{h} u^3 vs \rangle + \langle \tilde{h} uv^3 s \rangle), \tag{67}$$

and again by trying to write the differential eqautions for $\langle \tilde{h} u^3 vs \rangle$ and $\langle \tilde{h} uv^3 s \rangle$ we obtain

$$\begin{aligned}
\frac{d}{dt}\langle \tilde{h} u^3 vs \rangle &= 0 \\
\frac{d}{dt}\langle \tilde{h} uv^3 s \rangle &= 0
\end{aligned} \tag{68}$$

which results in $\langle \tilde{h} u^3 vs \rangle = \langle \tilde{h} uv^3 s \rangle = 0$, therefore $\langle \tilde{h}^2 uvs \rangle = 0$. Now $\langle \tilde{h} u^4 \rangle$, $\langle \tilde{h} v^4 \rangle$ and $\langle \tilde{h} u^2 v^2 \rangle$ should be found. The relating differential equation for $\langle \tilde{h} u^4 + \tilde{h} v^4 \rangle$ is

$$\begin{aligned}
&\frac{d}{dt}(\langle \tilde{h} u^4 \rangle + \langle \tilde{h} v^4 \rangle) \\
&\quad = -\gamma(t)(\langle u^4 + v^4 \rangle) - \frac{\alpha}{2}(\langle u^6 + v^6 \rangle) - \frac{\alpha}{2}(\langle u^4 v^2 + u^2 v^4 \rangle) \\
&\qquad - \alpha(\langle \tilde{h} u^4 w + \tilde{h} v^4 w \rangle) - \alpha(\langle \tilde{h} u^4 q + \tilde{h} v^4 q \rangle) - 12 k_{xx}(0, 0)(\langle \tilde{h} u^2 + \tilde{h} v^2 \rangle).
\end{aligned} \tag{69}$$

As before the following identities are held

$$\begin{aligned}
\langle \tilde{h} u^4 w \rangle &= -\tfrac{1}{5}\langle u^6 \rangle \\
\langle \tilde{h} u^4 q \rangle &= -\langle u^4 v^2 \rangle \\
\langle \tilde{h} v^4 q \rangle &= -\tfrac{1}{5}\langle v^6 \rangle \\
\langle \tilde{h} v^4 w \rangle &= -\langle u^2 v^4 \rangle ,
\end{aligned} \tag{70}$$

so we have

$$\frac{d}{dt}((\langle \tilde{h}u^4\rangle + \langle \tilde{h}v^4\rangle) = -\gamma(t)((\langle u^4+v^4\rangle) - \frac{3\alpha}{10}((\langle u^6+v^6\rangle) + \frac{\alpha}{2}((\langle u^4v^2+u^2v^4\rangle)$$
$$-12k_{xx}(0,0)((\langle \tilde{h}u^2+\tilde{h}v^2\rangle). \quad (71)$$

Substituting the expressions for $\gamma(t)$,$\langle u^4\rangle$, $\langle v^4\rangle$, $\langle u^6\rangle$, $\langle v^6\rangle$, $\langle u^4v^2\rangle$, $\langle u^2v^4\rangle$, and $(\langle \tilde{h}u^2+\tilde{h}v^2\rangle)$ we find

$$\langle \tilde{h}u^4+\tilde{h}v^4\rangle = 32\alpha k_{xx}(0,0)^3\, t^4. \quad (72)$$

The corresponding differential equation for $\langle \tilde{h}u^2v^2\rangle$ is

$$\frac{d}{dt}((\langle \tilde{h}u^2v^2\rangle) = -\gamma(t)((\langle u^2v^2\rangle) - \frac{\alpha}{2}((\langle u^4v^2+u^2v^4\rangle)$$
$$-\alpha((\langle \tilde{h}u^2v^2w+\tilde{h}u^2v^2q\rangle) - 2k_{xx}(0,0)((\langle \tilde{h}u^2+\tilde{h}v^2\rangle). \quad (73)$$

As before the following identities are held

$$\langle \tilde{h}u^2v^2w\rangle = -\tfrac{1}{3}\langle u^4v^2\rangle$$
$$\langle \tilde{h}u^2v^2q\rangle = -\tfrac{1}{3}\langle u^2v^4\rangle, \quad (74)$$

then by inserting the known moments we find

$$\langle \tilde{h}u^2v^2\rangle = \tfrac{16}{3}\,\alpha k_{xx}^3(0,0)\, t^4. \quad (75)$$

The above results get

$$\langle \tilde{h}^2u^2\rangle + \langle \tilde{h}^2v^2\rangle = -\tfrac{8}{3}\,\alpha^2k_{xx}^3(0,0)\, t^5 - 8k(0,0)\, k_{xx}(0,0)\, t^2, \quad (76)$$

which finally results in

$$\langle \tilde{h}^3\rangle = -\tfrac{48}{45}\,\alpha^3k_{xx}^3(0,0)\, t^6. \quad (77)$$

By continuing the above proccedure all the $\langle \tilde{h}^n\rangle$ moments can be derived. Some of these moments are listed bellow

$$\langle \tilde{h}^4\rangle = -\tfrac{44}{35}\,\alpha^4k_{xx}^4(0,0)\, t^8 - 8\alpha^2k(0,0)\, k_{xx}^2(0,0)\, t^5 + 12k^2(0,0)\, t^2 \quad (78)$$
$$\langle \tilde{h}^5\rangle = -\tfrac{1216}{945}\,\alpha^5k_{xx}^5(0,0)\, t^{10} - \tfrac{64}{3}\,\alpha^3k(0,0)\, k_{xx}^3(0,0)\, t^7, \quad (79)$$

which are the same as the results that we derived by expanding the generating function. Results of this appendix show that any moment containing the first power of s vanishes. Indeed, one can prove the following identity

$$\langle s e^{-i(\lambda \tilde{h} + \mu_1 u + \mu_2 v)} \rangle = 0. \tag{80}$$

By expanding the exponential in the above expression one finds

$$\langle s e^{-i(\lambda \tilde{h} + \mu_1 u + \mu_2 v)} \rangle = \sum_{n,\, m,\, p} \frac{i^{(n+m+p)} \lambda^n \mu_1^m \mu_2^p}{n!m!p!} \langle s \tilde{h}^n u^m v^p \rangle. \tag{81}$$

Now setting $n_3 = n_5 = 0$ and $n_4 = 1$ in Eq. (28) we get the following equation for $\langle s \tilde{h}^n u^m v^p \rangle$

$$\begin{aligned} \frac{d}{dt} \langle \tilde{h}^n u^m v^p s \rangle \\ = -n\gamma(t) \langle \tilde{h}^{n-1} u^m v^p s \rangle - \frac{\alpha n}{2} \langle \tilde{h}^{n-1} u^{m+2} v^p s \rangle - \frac{\alpha n}{2} \langle \tilde{h}^{n-1} u^m v^{p+2} s \rangle \\ + n(n-1)\, k_{xx}(0,0) \langle \tilde{h}^{n-2} u^m v^p s \rangle - m(m-1)\, k_{xx}(0,0) \langle \tilde{h}^n u^{m-2} v^p s \rangle \\ - p(p-1)\, k_{xx}(0,0) \langle \tilde{h}^n u^m v^{p-2} s \rangle. \end{aligned} \tag{82}$$

Starting from Eqs. (36), (43), and (58) all the $\langle s \tilde{h}^n u^m v^p \rangle$ moments can be evaluated and it can be shown that, with flat initial condition, they are equal to zero. So we conclude $\langle s\Theta \rangle = \langle u_y \Theta \rangle = \langle v_x \Theta \rangle = 0$.

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